Math 3236 Statistical Theory

1/24/23

X_i form a random sample

with p.d.f

$$f(x;\lambda) = \lambda e^{-\lambda x}$$

$$\mathbb{E}(X_i) = \frac{1}{\lambda} \qquad Var(X_i) = \frac{1}{\lambda^2}$$

Notation $\underline{X} = (X_1 \dots X_N)$

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Estimator for λ.

Method of moment:

$$\hat{\lambda}_N = \cancel{\frac{N}{\sum_{i=1}^{N} X_i}} = \cancel{\frac{1}{\bar{X}}}$$

$$\hat{\lambda}_N \xrightarrow{P} \frac{1}{\mathbb{E}(X_i)} = \lambda$$

Bias

$$S = \sum_{i=1}^{N} X_i$$

$$f_S(s) = \frac{\lambda^N s^{N-1}}{(N-1)!} e^{-\lambda s}$$

(Gamma dist. par N, λ)

$$\mathbb{E}(\hat{\lambda}_N) = \frac{N}{N-1} \lambda$$

$$\hat{\hat{\lambda}}_N = \frac{N-1}{N} \hat{\lambda}_N = N-1 \Big/ \sum_i X_i = N-1 \Big/ S$$

$\hat{\hat{\lambda}}_N$ is unbiased

$$\hat{\hat{\lambda}}_N \xrightarrow{P} \lambda$$

$$\mathbb{E}(1/S^2) = \frac{\lambda^2}{(N-1)(N-2)}$$

$$\mathrm{Var}(\hat{\hat{\lambda}}_N) = \frac{\lambda^2}{N-2}$$

You call an estimator The

MVUE Minimum variance unbiased estimator.

If I call $T_N = 1/S_N$

$$f_{T_N}(t) = \frac{\lambda^N t^{-(N+1)}}{(N-1)!} e^{-\lambda/t}$$

$$f_{\hat{\lambda}_N}(\ell) = \lambda^N \frac{N^{(N+1)}}{N!} \ell^{-(N+1)} e^{-\lambda N/\ell}$$

CLT

$$\lambda\sqrt{N}\left(\bar{X} - \frac{1}{\lambda}\right) \Rightarrow N(0,1)$$

using $\alpha(x) = \frac{1}{x}$

$$\frac{\lambda\sqrt{N}}{\lambda^2}\left(\frac{1}{\bar{X}} - \lambda\right) \Rightarrow N(0,1)$$

$$\hat{\lambda} \simeq N\left(\lambda, \frac{\lambda^2}{N}\right)$$

$$\hat{\lambda}_N(\underline{X})$$

$\underline{X}$ is the random sample

$\underline{x}$ is a realization.

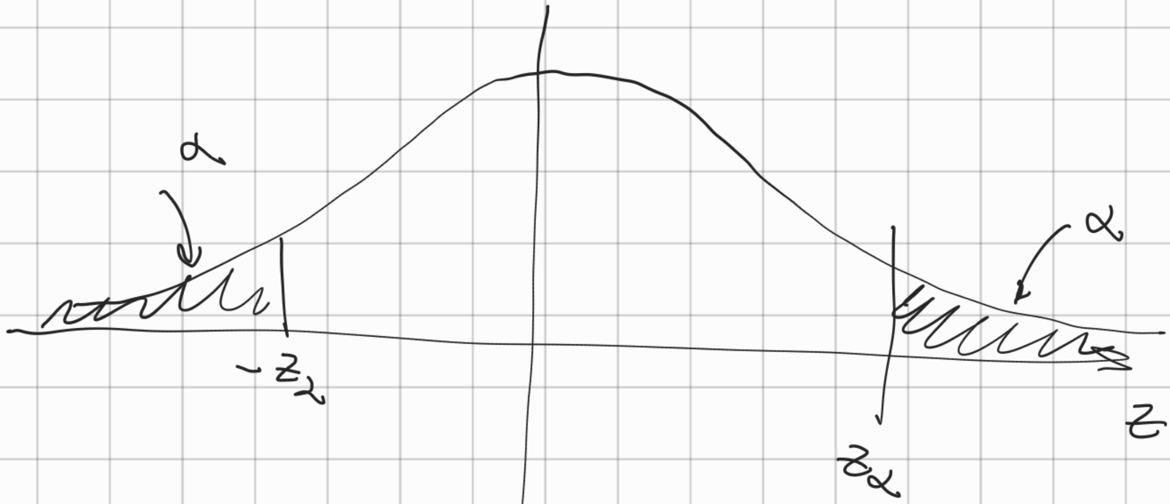
$\hat{\lambda}_N(\underline{X})$ is a r.v.
estimator

$\hat{\lambda}_N(\underline{x})$ estimate.

$$Z = \frac{\hat{\lambda}_N(\underline{X}) - \lambda}{\lambda / \sqrt{N}} \simeq N(0,1)$$

z_α α critical value

$$\mathbb{P}(Z > \alpha) = \alpha$$



$$\boxed{\Phi(-z_\alpha) = \alpha}$$

$$\mathbb{P}\left(-z_{\alpha/2} \le Z \le z_{\alpha/2}\right) = (1-\alpha)$$

$$\mathbb{P}\left(\hat{\lambda}(\underline{X}) - z_\alpha \frac{\lambda}{\sqrt{N}} \le \lambda \le \hat{\lambda}(\underline{X}) + \frac{\lambda z_\alpha}{\sqrt{N}}\right) = (1-\alpha)$$

C. I. with confidence $(1-\alpha)100\%$.

Gamma dist.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-\alpha x} \, dx$$

Prop.:

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$$

$$\Gamma(n) = (n-1)! \qquad n \text{ integer}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\lim_{x \to \infty} \frac{(2\pi)^{1/2} x^{x-\frac{1}{2}} e^{-x}}{\Gamma(x)} = 1$$

Stirling formula.

The p.d.f. of a $\Gamma(\alpha, \beta)$ r.v

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

$$\mathbb{E}(X) = \frac{\alpha}{\beta} \qquad V(X) = \frac{\alpha}{\beta^2}$$

$$\psi(t) = \mathbb{E}(e^{tX}) =$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} x^{\alpha-1} e^{-\beta x} \, dx =$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int x^{\alpha-1} e^{-(\beta-t)x} \, dx =$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta-t)^\alpha} = \left(\frac{\beta}{\beta-t}\right)^\alpha$$

$$\mathbb{E}(X^k) = \frac{d^k}{dt^k} \psi(t) \Big|_{t=0}$$

If $X_1 \sim \Gamma(\alpha_1, \beta)$ and

$X_2 \sim \Gamma(\alpha_2, \beta)$ ind.

$X_1 + X_2$

$$\psi_{X_1}(t) = \left(\frac{\beta}{\beta-t}\right)^{\alpha_1}$$

$$\psi_{X_2}(t) = \left(\frac{\beta}{\beta-t}\right)^{\alpha_2}$$

$$\psi_{X_1+X_2}(t) = \psi_{X_1}(t)\,\psi_{X_2}(t) =$$

$$= \left(\frac{\beta}{\beta - t}\right)^{\alpha_1+\alpha_2}$$

$$X_1 + X_2 \simeq \Gamma(\alpha_1 + \alpha_2, \beta)$$

if X is exp with par λ

$$X \simeq \Gamma(1, \lambda)$$

$$f_X(x) = \lambda e^{-\lambda x} = \frac{\lambda^1 x^0}{0!} e^{-\lambda x}$$

X_1 and X_2 are exp with par λ

$$X_1 + X_2 \simeq \Gamma(2, \lambda)$$

$$\sum_{i=1}^{N} X_i \simeq \Gamma(N, \lambda)$$

Beta distribution and

Bayes updates.

Coin flip.

p is unknown.

$X_1 \dots X_N$ is a random sample

$$\bar{X} = \frac{1}{N} \sum_i X_i$$

$$\mathbb{E}(X_i) = p \implies \hat{p} = \bar{X}$$

$$\mathbb{E}(\bar{X}) = p$$

Bayesian

Prior knowledge on p.

$$f(p) = 1 \qquad 0 \leq p \leq 1$$

Now I flip my coin and I

see $H = 1$

$$f(p \mid 1) \quad ??$$

$$f(0 \mid 1) = 0$$

$$f(p \mid 1) = \frac{\mathbb{P}(1 \mid p)\, f(p)}{\mathbb{P}(1)}$$

$$= \frac{\mathbb{P}(1 \mid p)\, f(p)}{\int_0^1 \mathbb{P}(1 \mid q)\, f(q)\, dq}$$

$$\mathbb{P}(1) = \int_0^1 \mathbb{P}(1 \mid p)\, f(p)\, dp$$

$$\mathbb{P}(1 \mid p) = p$$

$$f(p \mid 1) = \frac{\mathbb{P}(1 \mid p)\, f(p)}{\mathbb{P}(1)}$$

$$= \frac{p \cdot 1}{\int_0^1 p\, dp} = 2p$$

Posterior is

$$f(p \mid 1) = 2p$$

Observation

$f(p \mid 1)$ is a p.d.f.

$\mathbb{P}(1)$ is a number

$$f(p \mid 0) = 2(1-p)$$

$$f(p \mid 0,1) = \frac{p(1-p)}{\int_0^1 p(1-p)\,dp} \quad ??$$

$$= \frac{\mathbb{P}(0,1 \mid p)\, f(p)}{\mathbb{P}(0,1)}$$